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A COMPARISON OF CERTAIN MATRIX-BASED TECHNIQUES FOR SOLVING SYSTEMS OF EQUATIONS

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ABSTRACT

We considered Gaussian Elimination and inversion techniques for solving systems of linear equations. These equations feature predominantly in Science, Engineering and the social sciences. Both methods use specific matrix operations (elementary row operations) to obtain exact solutions to systems of equations. A review of the methods was presented, and an efficient, reliable and dependable technique was proposed to enhance effective teaching and learning of Mathematics. The comparison was made based on the number of operations visas versus the number of variables. The paper concludes that Gaussian Elimination was significantly more efficient and reliable when the number of variables increased.

KEYWORDS

Gaussian elimination, equivalence matrices, unique solution, system of equation

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INTRODUCTION

Systems of linear equations appear in many problems in Science, technology and engineering. The search for the most convenient solution technique to a system of linear equations has been going on for a long time, given the significant role these problems play in various fields of study. A system of equations is a set or collection of equations solved together. The collection of linear equations is termed a system of linear equations. They are often based on some set of variables. Various methods have been evolved to solve linear equations, but the best method is yet to be proposed for solving the system of linear equations (Jamil, 2012). Different mathematicians propose various methods in search of speed and accuracy. However, speed is essential for solving linear equations with large computation volumes. Jeremy *et al*. (1996).

Methods of solution to a system of linear equations are divided into two categories. Direct and Indirect. Many researchers have investigated the solutions of systems of linear equations through direct and indirect methods (Dass & Rama, 2010; Dafchahi, 2010). Systems of linear equations exist in many areas, either directly in modelling physical situations or indirectly in the numerical solutions of other mathematical models. The application of systems of linear equations occurs in virtually all areas of the Physical, Biological and Social Sciences. Linear systems are at the heart of numerical solutions to optimization problems, systems of non-linear equations, partial differential equations, etc.

Given a system of linear equations

$$\begin{matrix} a_{11}x_1 & a_{12}x_2 & \dots & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & a_{22}x_2 & \dots & a_{2n}x_n & = & b_2 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ a_{n1}x_1 & a_{n2}x_2 & \cdots & a_{nn}x_n & = & b_n \end{matrix}$$

Where $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \boldsymbol{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \; and \; \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

A is the matrix of coefficient, $\boldsymbol{X}$ is the matrix of unknown, and $\boldsymbol{b}$ is the matrix of the constant associated with each of the sets in the linear system.

The system can be represented as

$$\boldsymbol{AX} = \boldsymbol{b} \tag{1}$$

The solution to (1) has played a significant role in a wide area of mathematics. These solutions are obtained either by analytical techniques or numerical procedures. Matrix analysis has played a significant role in solving (1). These include using the crammers' rule, Gaussian Elimination, inversion technique, and LU factorization. A search for an appropriate, practical teaching and learning method must be considered, given the decline in student performance in mathematics and other science and engineering fields. Many researchers have sought an appropriate technique by comparing different methods of solution (Thirumurugan, 2014).

Smith and Powell (2011) presented an alternative method to Gauss-Jordan Elimination. A modification was proposed in their work, and a few limitations were listed. Yadanar *et al*. (2014) compared the performance of Gaussian Elimination and the Gauss-Jordan method. Suriya *et al.* (2015) compared two direct methods, Gaussian elimination and Gauss Jordan method. Their work analyzed the performance of each method on the basis of execution time. Ogunlade (2018) studied three iterative methods: Successive-Over Relaxation, the Gauss-Seidel and the Jacobi technique. Number of iteration and storage were used as bases for comparison. The work concluded that successive-over relaxation method could be considered more efficient out of the three iterative methods considered. Haoyu *et al.* (2021) studied three direct methods for solving systems of linear equation. In their work, advantages and disadvantages of each of variable elimination, Gaussian elimination and Cramer's rule were presented.

We present Gaussian Elimination and the matrix inversion techniques as reliable tools for solving systems of linear equations for enhanced teaching and learning of Mathematics. These two methods are based on specific operations on certain matrices, thereby presenting is a common ground for comparison. depend on elementary row operations on the augment

PROPOSITION

Inversion Technique **(**Larson & Falvo, 2009))

The proposed solution is based on the procedure below:

Given the system $AX = b$ where A^{-1}, the inverse of the matrix A exist,

multiplying (1) by A^{-1}, we have $A^{-1}AX = A^{-1}b$ (by associativity of matrix multiplication)

$$IX = A^{-1}b$$

$$X = A^{-1}b = K \quad (2)$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}$$

Gaussian Elimination

Gaussian Elimination is the systematic application of elementary row operations in a system of equations. It converts the linear system of equations to upper triangular form, from which the solution of an equation is determined. Gaussian Elimination is summarized in the steps mentioned below:

i. An augmented matrix must be written for the system of linear equations.

ii. Transform A to upper triangular form using row operations on $(A|B)$. Diagonal elements may not be zero.

iii. Use back substitution to find the solution to the unknowns.

$$\begin{matrix} a_{11}x_1 & a_{12}x_2 & \dots & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & a_{22}x_2 & \dots & a_{2n}x_n & = & b_2 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ a_{n1}x_1 & a_{n2}x_2 & \cdots & a_{nn}x_n & = & b_n \end{matrix} \Rightarrow \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

We obtain an augmented matrix $(A|b.)$

$$\left(\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix}\middle| \begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{matrix}\right)$$

Using elementary operation, we reduce the augmented matrix to an upper triangular matrix below,

$$\left(\begin{matrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & & \\ \vdots & \vdots & & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{matrix}\middle| \begin{matrix} a_{1(n+1)} \\ a_{2(n+1)} \\ a_{3(n+1)} \\ \vdots \\ a_{n(n+1)} \end{matrix}\right)$$

$$\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & & \\ \vdots & \vdots & & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \middle| \begin{matrix} a_{1(n+1)} \\ a_{2(n+1)} \\ a_{3(n+1)} \\ \vdots \\ a_{n(n+1)} \end{matrix}\right)$$

Table 1: Comparison through the number of operations

Number of variables	Number of operations	
	Inversion Technique	Gaussian Elimination
2	2	3
3	8	5
4	12	7
5	18	11
6	26	17

Theoretical examples

$$\begin{aligned} 3x - 3y + 4z &= 6 \\ 2x - 3y + 4x &= 5 \\ -y + z &= 1 \end{aligned} \Rightarrow \left(\begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{matrix} 6 \\ 5 \\ 1 \end{matrix}\right)$$

Using inversion technique:

$$\left\{\begin{matrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{matrix}\middle|\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right\} \sim \frac{1}{3}R_1 \left\{\begin{matrix} 1 & -1 & \frac{4}{3} \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{matrix}\middle|\begin{matrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right\} \sim R_2 - 2R_1 \left\{\begin{matrix} 1 & -1 & \frac{4}{3} \\ 0 & -1 & \frac{4}{3} \\ 0 & -1 & 1 \end{matrix}\middle|\begin{matrix} \frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right\}$$

$$\sim -R_2 \left\{\begin{matrix} 1 & -1 & \frac{4}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & -1 & 1 \end{matrix}\middle|\begin{matrix} \frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & -1 & 0 \\ 0 & 0 & 1 \end{matrix}\right\} \sim \begin{matrix} R_1 + R_2 \\ \\ R_3 + R_2 \end{matrix} \left\{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & -\frac{1}{3} \end{matrix}\middle|\begin{matrix} 1 & -1 & 0 \\ \frac{2}{3} & -1 & 1 \\ \frac{2}{3} & -1 & 1 \end{matrix}\right\}$$

$$\underset{-3R_3}{\sim} \left\{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 1 \end{matrix}\middle|\begin{matrix} 1 & -1 & 0 \\ \frac{2}{3} & -1 & 1 \\ 2 & 3 & -3 \end{matrix}\right\} \sim \begin{matrix} R_2 + \frac{4}{3}R_3 \\ -3R_3 \end{matrix} \left\{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\middle|\begin{matrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{matrix}\right\}$$

Therefore: $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

$$hence, \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} \times \begin{pmatrix} 6 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Using Gaussian Elimination

$$\begin{aligned} 3x - 3y + 4z &= 6 \\ 2x - 3y + 4x &= 5 \\ -y + z &= 1 \end{aligned} \Rightarrow \left(\begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}\middle| \begin{matrix} 6 \\ 5 \\ 1 \end{matrix}\right)$$

$$\left(\begin{matrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{matrix}\middle|\begin{matrix} 6 \\ 5 \\ 1 \end{matrix}\right) \Rightarrow R_2 - \tfrac{2}{3}R_1 \left(\begin{matrix} 3 & -3 & 4 \\ 0 & -1 & \frac{4}{3} \\ 0 & -1 & 1 \end{matrix}\middle|\begin{matrix} 6 \\ 1 \\ 1 \end{matrix}\right) \Rightarrow R_3 - R_2 \left(\begin{matrix} 3 & -3 & 4 \\ 0 & -1 & \frac{4}{3} \\ 0 & 0 & \frac{-1}{3} \end{matrix}\middle|\begin{matrix} 6 \\ 1 \\ 0 \end{matrix}\right)$$

Applying back substitution,

$z = 0, -y + \frac{4}{3}(0) = 1, \Rightarrow\ y = -1,$

$3x - 3(-1) + 4(0) = 6, \Rightarrow\ x = 1$

CONCLUSION

In this paper, a comparison was made between two matrix-based techniques for solving a system of linear equations. The Gaussian elimination technique was observed to be the most efficient and requires less effort when the elementary operations are correctly understood. Although the inversion technique was efficient with two variables, the technique is more cumbersome as the number of variables increases. This comparison was done based on the number of operations to determine efficiency. The work may be extended to involve using computer algorithms to solve the system in order to determine efficiency based on time and memory requirement. It is hoped that teaching and learning of systems of linear equations will be enhanced if the appropriate technique is used. This will go a long way to stimulating interest and encouraging learning.

REFERENCES

Dafchahi. (2010). A new refinement of Jacobi Method for solution of linear system of equations. *Journal of Computer and Mathematical Science, 3*, 819-827.

Dass, H. K., & Rama, V. (2010). Numerical solution of linear equations solved by direct and iterative methods. *Numerical Analysis, 3*, 566-580.

Haoyu, L., Simeng, W., & Ningyun, X. (2021). Three methods of solving systems of linear equations: Comparing the advantages and disadvantages. *Journal of Physics Conferences Series*.

Jamil, N. (2012). Direct and indirect solvers for linear system equations. *International Journal of Emerging Sciences, 2*(2), 310-322.

Jeremy, J., Croz, D., & Nicholas, J. H. (1992). Stability of methods for matrix inversion. *IMA Journal of Numerical Analysis, 12*, 1-19.

Larson, R., & Falvo, D.C. (2009). *Elementary Linear Algebra* (6th Ed). Houghton Mifflin Harcourt Publishing Company.

Ogunlade, T. O. (2018). On the direct and indirect methods of solving systems of linear equations. *Scholars Journal of Physics, Mathematics and Statistics, 5*(6), 334-341.

Smith, L., & Powell, J. (2011). An alternative method to Gauss-Jordan elimination: Minimizing fraction arithmetic. *The Mathematics Educator*.

Suriya, G., Syeda, R.A., Rabia, K., Nargis, M., & Memoona, K. (2015). System of linear equations: Gaussian elimination. *Global Journal of Computer Science and Technology, 15*(5), 23-26.

Thirumurugan, K. (2014). A new method to compute the adjoint and inverse of a 3×3 non-singular matrices. *International journal of Mathematics and Statistics Invention, 2*(10), 52-55.

Tian, N., Guo, L., Ren, M., & Ai, C. (2014). Implementing the matrix inversion by Gauss-Jordan method with CUDA. *Wireless Algorithms, Systems and Applications WASA, 1*, 84-91.

Yadanar, M., Lai, L. Win, K. (2014). Performance comparison of Gaussian elimination and Gauss Jordan. *International Journal of Computer & Communication Engineering Research (IJCCER), 2*(2), 1-4.