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AN ALTERNATIVE APPROACH FOR SOLVING A SYSTEM OF LIINEAR EQUATIONS USING THE MATRIX INVERSION TECHNIQUE

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ABSTRACT

The matrix inversion technique is one of the tools for solving a system of linear equations. An alternative approach to computing the inverse of square matrices is proposed here using Cramer's rule in solving a linear equation system. The inverse is obtained as a coefficient of a catalytic column vector of a supposed solution of a linear system of equation $AX = b$. This inverse is used to obtain a unique solution of systems of linear equations of various sizes of unknown with practical illustrations shown.

KEYWORDS
Cramer’s rule, inverse, system of equation

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INTRODUCTION

The search for the most convenient technique for linear equations has been going on at the right time, given the significant role systems of linear equations play in various fields of study. A system of m linear equations in n variables is a set of m equations, each of which is linear in the same n variables.. Various methods have been evolved to solve linear equations, but the best method is yet to be proposed for solving a system of linear equations (Jamil, 2012).

Different mathematicians propose various methods based on efficiency and accuracy. However, speed is essential for solving linear equations where the computation volume is so large.

Solutions to system of linear equation can be obtained either by direct or indirect approach. The direct approach employs the techniques of Linear Algebra to find the values of the variables which satisfies the sets of equation. This method attempts to calculate an exact solution in a finite number of operations. On the other hand, the indirect method uses Numerical techniques to approximate the solutions based on certain methods of iterations. Many researchers have investigated the solutions of systems of linear equations through direct and indirect methods (Dass & Rama, 2010; Dafchahi, 2010).

Haoyu *et a*l (2021) studied three direct methods for solving systems of linear equation. In their work, advantages and disadvantages of each of variable elimination, Gaussian elimination and Cramer's rule were presented. Suriya *et al* (2015) compared two direct methods, Gaussian elimination and Gauss Jordan method. Their work analyzed the performance of each method on the basis of execution time. Kalambi (2008) studied three main iterative methods for solving linear equation: These are Successive-Over Relaxation, the Gauss-Seidel and the Jacobi technique. Systems of linear equations exist in many areas, either directly in modelling physical situations or indirectly in the numerical solutions of other mathematical models. The application of systems of linear equations occurs in virtually all areas of Physical, Biological and Social sciences. Linear systems are at the heart of numerical solutions to optimization problems, systems of non-linear equations, partial differential equations, etc.

Given a system of linear equations

$$\begin{matrix} a_{11}x_1 & a_{12}x_2 & \dots & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & a_{22}x_2 & \dots & a_{2n}x_n & = & b_2 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ a_{n1}x_1 & a_{n2}x_2 & \cdots & a_{nn}x_n & = & b_n \end{matrix}$$

Where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \boldsymbol{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} and\ \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

A is the matrix of coefficient, ***X*** is the matrix of unknown, and ***b*** is the matrix of the constant associated with each of the sets in the linear system.

The system can be represented as

$$AX = b \quad (1)$$

The solution to (1) has played a significant role in a wide area of mathematics, obtained either by analytical techniques or numerical procedure.

We consider an aspect of the analytical technique, the matrix inversion technique and propose a new approach to obtaining the solution to the system of linear equations given as (1).

MAIN RESULT

The proposed solution is based on the procedure below:

Given the system $AX = b$ where A^{-1}, the inverse of the matrix A exists, multiplying (1) by A^{-1}, we have $A^{-1}AX = A^{-1}b$(by associativity of matrix multiplication)

$$IX = A^{-1}b$$

$$X = A^{-1}b \quad (2)$$

Assuming that system (1) has a unique solution, we aim to obtain A^{-1} and subsequently use it to compute X, as shown in (2), where X is the matrix of unknowns in the system (1).

Several methods have been developed for the inversion of matrices, as discussed in Tian *et al*. (2014), Smith and Powell (2011), Thirumurugan (2014), Jeremy *et al*. (1991) and John (2019).

Existing methods mainly rely on the use of cofactors and adjoints of the matrix A. These methods place a high demand on cost of computation, significantly when the number of unknowns increases. The researchers intend to adopt a procedure proposed in John (2019) to obtain the inverse of A given in (1).

Proposition 1 Cramer's rule

Let $AX = b$ be a linear system of equations defined as (i) with unknown variables $x_1, x_2, \dots x_n$and $a_{11}, a_{12}, \dots a_n$d denotes the entries of the coefficient matrix A. Let$\det(A) \neq 0$, be the determinant of A, then the solution $x_1, x_2, \dots x_n$ of (1) is given by $x_i = \frac{\det(A_i)}{\det(A)}$for $i = 1,2,3, \dots n$.

Proof

Given a system of linear equations:

$$\begin{array}{ccccccc} a_{11}x_1 & a_{12}x_2 & \dots & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & a_{22}x_2 & \dots & a_{2n}x_n & = & b_2 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ a_{n1}x_1 & a_{n2}x_2 & \cdots & a_{nn}x_n & = & b_n \end{array}$$

If we write the system as $AX = b$, then, provided $\det(A) \neq 0$, the solution can be written as:

$$X = A^{-1}b = \frac{1}{\det(A)}(adjA)b = \frac{1}{\det(A)}C^T b$$

where C^T is the transpose of the matrix of cofactors of A.

If $X = (x_1, x_2, \dots x_n)^T$ and $b = (b_1, b_2, \dots b_n)^T$, the i^{th} element of X is given by

$$x_i = \frac{1}{\det(A)}(c_{1i}b_1) + (c_{2i}b_2) + \cdots + (c_{ni}b_n) \text{ for } i = 1,2,3, \dots n.$$

This is simply the expansion of $\det(A_i)$ in terms of the elements of its i^{th} column, where A_i is the matrix obtained from A by replacing the elements of the i^{th} column with the elements of b.

This has established that $x_i = \frac{\det(A_i)}{\det(A)}$ for $i = 1,2,3, \dots n$

Proposition 2 (John, 2019)

Let x_k $(k = 1,2 \dots n)$ be the solution of (1) obtained by Crammer's rule, then

(i) $x_k = \sum_{j=1}^{n} D_{jk} b_j$

(ii) $x = \Delta b$ where $\Delta = \begin{pmatrix} D_{11} & D_{12} & \cdots & D_{1n} \\ D_{21} & D_{22} & \cdots & D_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ D_{n1} & D_{n2} & \cdots & D_{nn} \end{pmatrix}$

Proof (See John 2019)

Proposition 3 (John, 2019)

The matrix D in Proposition 2 is the inverse of A.

Proof

$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ be the solution of (1) then,

$$x = A^{-1}b = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} D_{11} & D_{12} & \cdots & D_{1n} \\ D_{21} & D_{22} & \cdots & D_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ D_{n1} & D_{n2} & \cdots & D_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \Delta b$$

Therefore, $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}^{-1} = \begin{pmatrix} D_{11} & D_{12} & \cdots & D_{1n} \\ D_{21} & D_{22} & \cdots & D_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ D_{n1} & D_{n2} & \cdots & D_{nn} \end{pmatrix}, i.e. A^{-1} = \Delta$

Proposition 4 (Larson & Falvo, 2009))

The solution of a system of linear equation $AX = b$ can be obtained as $X = \Delta b$

where $\Delta = A^{-1}$.

Proof

Let $AX = b$ be a system of linear equations with n-unknown

To obtain Δ, let $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ $and\ b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ be column vectors such that the system (1) is consistent. Since our interest is to find Δ, choosing the entries of b as unknown constraints does not affect the outcome of the computation since b is only a factor of $A^{-1}b$.

Multiplying (1) by Δ yields $\Delta AX = \Delta b$

But $\Delta A = I, (since\Delta = A^{-1})$

$IX = \Delta b$

$X = \Delta b$. hence the proof.

APPLICATION

In this section, we compute Δ using the procedure given by (John, 2019) and obtain the solution of (1) for various unknowns where A is any non-singular square matrix with $|A| \neq 0$.

Example 1

Find the solution of the system of linear equations given as $\begin{matrix} 3x_1 + 5x_2 = 1 \\ x_1 - 4x_2 = 6 \end{matrix}$

Solution

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, then $AX = bgives \begin{pmatrix} 3 & 5 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$$x_1 = \frac{\Delta_1}{\Delta_0} = \frac{\begin{vmatrix} b_1 & 5 \\ b_2 & -4 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 1 & -4 \end{vmatrix}} = \frac{-4b_1 - 5b_2}{-17},$$

$$x_2 = \frac{\Delta_2}{\Delta_0} = \frac{\begin{vmatrix} 3 & b_1 \\ 1 & b_2 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 1 & -4 \end{vmatrix}} = \frac{3b_2 - b_1}{-17},$$

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{-1}{17} \begin{pmatrix} -4b_1 - 5b_2 \\ 3b_2 - b_1 \end{pmatrix} = \frac{-1}{17} \begin{pmatrix} -4 & -5 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$\therefore \Delta = \frac{-1}{17}\begin{pmatrix} -4 & -5 \\ -1 & 3 \end{pmatrix}$

Applying (2)

$X = \Delta b$

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{-1}{17}\begin{pmatrix} -4 & -5 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 6 \end{pmatrix} = \frac{-1}{17}\begin{pmatrix} -4-30 \\ -1+18 \end{pmatrix} = \frac{-1}{17}\begin{pmatrix} -34 \\ 17 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Hence, $x_1 = 2$ and$x_2 = -1$.

Example 2

Find the solution of the system of the linear equation given as $\begin{matrix} 4x_1 + 2x_2 - x_3 = 9 \\ x_1 - x_2 + 3x_3 = -4 \\ 2x_1 + x_3 = 1 \end{matrix}$

Solution

The system can be reduced to $\begin{pmatrix} 4 & 2 & -1 \\ 1 & -1 & 3 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 9 \\ -4 \\ 1 \end{pmatrix}$

We proceed to find Δ the inverse of $A = \begin{pmatrix} 4 & 2 & -1 \\ 1 & -1 & 3 \\ 2 & 0 & 1 \end{pmatrix}$

Then,

$$x_1 = \frac{\Delta_1}{\Delta_0} = \frac{\begin{vmatrix} b_1 & 2 & -1 \\ b_2 & -1 & 3 \\ b_3 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 4 & 2 & -1 \\ 1 & -1 & 3 \\ 2 & 0 & 1 \end{vmatrix}} = \frac{-b_1 - 2b_2 + 5b_3}{4},$$

$$x_2 = \frac{\Delta_2}{\Delta_0} = \frac{\begin{vmatrix} 4 & b_1 & -1 \\ 1 & b_2 & 3 \\ 2 & b_3 & 1 \end{vmatrix}}{\begin{vmatrix} 4 & 2 & -1 \\ 1 & -1 & 3 \\ 2 & 0 & 1 \end{vmatrix}} = \frac{5b_1 + 6b_2 - 13b_3}{4}$$

$$x_3 = \frac{\Delta_3}{\Delta_0} = \frac{\begin{vmatrix} 4 & 2 & b_1 \\ 1 & -1 & b_2 \\ 2 & 0 & b_3 \end{vmatrix}}{\begin{vmatrix} 4 & 2 & -1 \\ 1 & -1 & 3 \\ 2 & 0 & 1 \end{vmatrix}} = \frac{2b_1 + 4b_2 - 6b_3}{4}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -1 & -2 & 5 \\ 5 & 6 & -13 \\ 2 & 4 & -6 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

$$\therefore \Delta = \frac{1}{4}\begin{pmatrix} -1 & -2 & 5 \\ 5 & 6 & -13 \\ 2 & 4 & -6 \end{pmatrix}$$

Applying (2)

$$X = \Delta b$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -1 & -2 & 5 \\ 5 & 6 & -13 \\ 2 & 4 & -6 \end{pmatrix}\begin{pmatrix} 9 \\ -4 \\ 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -9+8+5 \\ 45-24-13 \\ 18-16-6 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 4 \\ 8 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Hence, $x_1 = 1, x_2 = 2$ and$x_3 = -1$.

Example 3

Solve the system of the linear equation given as
$$\begin{gathered} 2x_1 + x_2 + 2x_3 + x_4 = 6 \\ 6x_1 - 6x_2 + 6x_3 + 12x_4 = 36 \\ 4x_1 + 3x_2 + 3x_3 - 3x_4 = -1 \\ 2x_1 + 2x_2 - x_3 + x_4 = 10 \end{gathered}$$

Solution

The system can be reduced to $\begin{pmatrix} 2 & 1 & 2 & 1 \\ 6 & -6 & 6 & 12 \\ 4 & 3 & 3 & -3 \\ 2 & 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 6 \\ 36 \\ -1 \\ 10 \end{pmatrix}$

We proceed to find Δ the inverse of $A = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 6 & -6 & 6 & 12 \\ 4 & 3 & 3 & -3 \\ 2 & 2 & -1 & 1 \end{pmatrix}$

Then,

$$x_1 = \frac{\Delta_1}{\Delta_0} = \frac{\begin{vmatrix} b_1 & 1 & 2 & 1 \\ b_2 & -6 & 6 & 12 \\ b_3 & 3 & 3 & -3 \\ b_4 & 2 & -1 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 2 & 1 \\ 6 & -6 & 6 & 12 \\ 4 & 3 & 3 & -3 \\ 2 & 2 & -1 & 1 \end{vmatrix}} = \frac{-162b_1+27b_2+72b_3+54b_4}{234},$$

$$x_2 = \frac{\Delta_2}{\Delta_0} = \frac{\begin{vmatrix} 2 & b_1 & 2 & 1 \\ 6 & b_2 & 6 & 12 \\ 4 & b_3 & 3 & -3 \\ 2 & b_4 & -1 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 2 & 1 \\ 6 & -6 & 6 & 12 \\ 4 & 3 & 3 & -3 \\ 2 & 2 & -1 & 1 \end{vmatrix}} = \frac{180b_1-30b_2-54b_3+18b_4}{234},$$

$$x_3 = \frac{\Delta_3}{\Delta_0} = \frac{\begin{vmatrix} 2 & 1 & b_1 & 1 \\ 6 & -6 & b_2 & 12 \\ 4 & 3 & b_3 & -3 \\ 2 & 2 & b_4 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 2 & 1 \\ 6 & -6 & 6 & 12 \\ 4 & 3 & 3 & -3 \\ 2 & 2 & -1 & 1 \end{vmatrix}} = \frac{138b_1-10b_2-18b_3-72b_4}{234},$$

$$x_4 = \frac{\Delta_4}{\Delta_0} = \frac{\begin{vmatrix} 2 & 1 & 2 & b_1 \\ 6 & -6 & 6 & b_2 \\ 4 & 3 & 3 & b_3 \\ 2 & 2 & -1 & b_4 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 2 & 1 \\ 6 & -6 & 6 & 12 \\ 4 & 3 & 3 & -3 \\ 2 & 2 & -1 & 1 \end{vmatrix}} = \frac{102b_1-4b_2-54b_3+18b_4}{234},$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \frac{1}{234}\begin{pmatrix} -162 & 27 & 72 & 54 \\ 180 & -30 & -54 & 18 \\ 138 & -10 & -18 & -72 \\ 102 & -4 & -54 & 18 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix},$$

$$\therefore \Delta = \frac{1}{234}\begin{pmatrix} -162 & 27 & 72 & 54 \\ 180 & -30 & -54 & 18 \\ 138 & -10 & -18 & -72 \\ 102 & -4 & -54 & 18 \end{pmatrix}$$

Applying (2)

$$X = \Delta b$$

$$X=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\frac{1}{234}\begin{pmatrix}-162 & 27 & 72 & 54\\180 & -30 & -54 & 18\\138 & -10 & -18 & -72\\102 & -4 & -54 & 18\end{pmatrix}\begin{pmatrix}6\\36\\-1\\10\end{pmatrix}=\frac{1}{234}\begin{pmatrix}-972+952-72+540\\1080-1080+54+180\\828-360+18-720\\612-144+54+180\end{pmatrix}$$

$$=\frac{1}{234}\begin{pmatrix}468\\234\\-234\\702\end{pmatrix}=\begin{pmatrix}2\\1\\-1\\3\end{pmatrix}$$

Hence, $x_1=2, x_2=1$, $x_3=-1$ and $x_4=3$

CONCLUSION

This paper proposes an efficient approach based on the matrix inversion technique for solving a system of linear equations for n-unknown. The techniques are rapid, accessible, efficient, usable, and highly accurate. The new method creates opportunities to find other methods based on the inversion techniques for solving system of linear equation. This new approach is applicable to solve systems of linear equations of more considerable unknowns such as 5, 6, etc., as presented in the second part of the paper (next article).

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